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Chaos and Scaling in Classical Non-Abelian Gauge Fields

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Abstract

Without an ultraviolet cut-off, the time evolution of the classical Yang-Mills equations give rise to a never ending cascading of the modes towards the ultraviolet, and ergodic measures and dynamical averages, such as the spectrum of characteristic Lyapunov exponents (measures of temporal chaos) or spatial correlation functions, are ill defined. A lattice regularization (in space) provides an ultraviolet cut-off of the classical Yang-Mills theory, giving a possibility for the existence of ergodic measures and dynamical averages. We analyze in this investigation in particular the scaling behavior $\beta = d \log \lambda / d \log E$ of the principal Lyapunov exponent with the energy of the lattice system. A large body of recent literature claims a linear scaling relationship ($\beta = 1$) between the principal Lyapunov exponent and the average energy per lattice plaquette for the continuum limit of the lattice Yang-Mills equations. We question this result by providing rigorous upper bounds on the Lyapunov exponent for all energies, hence giving a non-positive exponent, $\beta \leq 0$, asymptotically for high energies, and we give plausible arguments for a scaling exponent close to $\beta \sim 1/4$ for low energies. We argue that the region of low energy is the region which comes closest to what could be termed a "continuum limit" for the classical lattice system.

The classical Yang-Mills equations for non-Abelian gauge fields provide an interesting class of dynamical systems for which non-linear self-coupling terms open up the possibility of chaotic behavior even in toy-models where the gauge fields are spatially homogeneous [1] - [6]. However, a qualitative important new dynamical feature comes into play when one considers the spatially inhomogeneous classical Yang-Mills equations: There is a never ending cascading of the dynamical degrees of freedom towards the ultraviolet, generated by the time evolution of the Yang-Mills equations as we shall discuss below. In spite of this "ultraviolet catastrophe", we note that the solutions are well behaved, in the sense that there are no "finite time blow up of singularities". For example, the class of classical Yang-Mills equations in 3+1 dimensions is known to have solutions in suitable Sobolev spaces, such that their norms do not blow up to infinity in finite time [7, 8]. Without an ultraviolet cut-off of the Yang-Mills field equations, it is however not possible - even when the fields are confined to a finite volume - to define equilibrium statistical mechanics, i.e. make reasonable sense out of infinite-time time averages of observables and correlation functions. The reason for this is that although the system has finite space volume, the phase space volume is still infinite. As is well-known in the case of electromagnetism, a reasonable definition of equilibrium leads to an ultraviolet catastrophe as seen in the Rayleigh-Jeans law for energy (per frequency) density. The arguments in the electromagnetic case are purely thermodynamic and do not involve the classical evolution of the fields. In fact, since the corresponding equations are linear the problems would not show up dynamically unless one couples the fields to charged particles. In contrast, the Yang-Mills equations for non-Abelian gauge fields include non-linear self-coupling terms which open up the possibility for a chaotic behavior in the classical evolution and in the non-homogeneous case it leads to the infinite cascade of energy from the long wavelength modes towards the ultraviolet. This is a natural dynamical interpretation of the ultraviolet catastrophe if one assumes that on average a trajectory should simulate a micro canonical ensemble where, in an equilibrium situation, all modes have the same energy on average. But due to the infinite phase space volume, this implies that this average energy per mode has to be zero (!), whence the cascade. We would like to add that this picture is well supported by the articles of Furusawa [9] and Wellner [10].

This tendency of the mode frequencies cascading towards the ultraviolet will completely dominate the qualitative behavior of the classical Yang-Mills equations, and the "ultraviolet catastrophe" has for some time been emphasized by us (H.B.N. and S.E.R., cf. e.g. discussion in [26]) as a major obstacle to simulate the classical continuum Yang-Mills fields in a numerical experiment over a long time span. (This obstacle has in our opinion received insufficient attention in the various studies attempting at discussing and modeling chaotic properties of spatially inhomogeneous classical Yang-Mills fields). There is no mechanism, within the classical equations, which prevents this never ending cascading of the modes towards the ultraviolet. Nature needs $\hbar$, the Planck constant, as an ultraviolet regulator. Indeed, both Abelian and non-Abelian gauge fields are implemented as quantum theories in Nature.

Here, however, we shall discuss the possibility of using a lattice cutoff in a purely classical

treatment to regularize the equations and giving sensible results in the limit when the lattice spacing a tends to zero. The model we consider is the Hamiltonian lattice formulation first introduced by Kogut and Susskind [13] (in a quantum context). There is still a cascading of modes towards the ultraviolet, i.e. towards the lattice cut-off, and this ultraviolet cascade will still dominate the dynamical evolution of smooth initial field configurations. However, in this Hamiltonian formulation on a large but finite lattice, the phase space is compact for any given energy and thus the system can reach an equilibrium state among the modes (a 'thermodynamic equilibrium'). The lattice regularization of the theory opens up for defining dynamic and thermodynamic properties, which are not defined in the classical Yang-Mills field theory without regularization. It could e.g. be ergodic (modulo constraints) with respect to the Liouville measure, in which case it makes sense to talk about its micro-canonical distribution and approximating this by looking at 'typical' classical trajectories. The fundamental assumption of thermodynamics asserts that on average the two approaches give the same result if we have a large system, and we may then naturally introduce correlation functions and possibly a correlation length ξ (measured in lattice units) of the system. One hopes to define a continuum theory if, by judicious choice of the parameters in the system, one obtains a physical correlation length in the limit when the lattice constant goes to zero, i.e.

$$\xi(a, E(a), ...) \times a \quad \rightarrow \quad \ell \neq 0 \ \ , \ \ \text{as } a \rightarrow 0. \tag{1}$$

In equation (1) the correlation length ξ in the lattice system is a function of lattice model parameters such as lattice spacing a, average energy density $E(a)$, etc. Condition (1) implies that the correlation length diverges when measured in lattice units and only if this is the case do we expect the lattice system to lose its memory of the underlying lattice structure. Condition (1) is, of course, just a necessary, not sufficient condition for the lattice system to approach the continuum theory. For example, the limiting system might still carry the symmetries of the lattice. We shall return in more detail to the important issue of what we could mean by a "continuum limit" of a classical lattice regularized non-Abelian gauge field theory.

The Kogut-Susskind Hamiltonian is obtained from the Wilson action, approximating the Yang-Mills equations, by keeping a lattice grid in space while letting time become continuous. For a derivation we refer to [13] and also to [14] from which we adapt our notation. We will restrict attention to the lattice gauge theory in 3+1 dimensions based on the gauge group $SU(2)$. One considers a finite size 3 dimensional discretized box, i.e. a finite lattice having N^3 points where nearest neighbor points are separated by a distance $a > 0$. The phase space corresponding to this is then a fibered space where the tangent manifold of the Lie-group $SU(2)$ is assigned to each of the links, $i \in \Lambda$, connecting nearest neighbor lattice points. More precisely, we have to each $i \in \Lambda$ associated a link variable $U_i \in SU(2)$ as well as its canonical momentum $P_i \in T_{U_i}SU(2)$. A point in the entire phase space will be denoted

$$\mathbf{X} = \{U_i, P_i\}_{i \in \Lambda} \ \in \ M = \prod_{\Lambda} T \ SU(2) \ . \tag{2}$$

The resulting Kogut-Susskind Hamiltonian generating the time evolution of the orbit $\mathbf{X}(t)$ can be written in the following way, where for simplicity we omit the coupling constant factor $2/g^2$ which anyway is arbitrary in a classical theory :

$$H(a,\mathbf{X}^{(a)}) = \frac{1}{a}\sum_{i\in\Lambda}\frac{1}{2}\mathrm{tr}(P_iP_i^\dagger) + \frac{1}{a}\sum_{\Box}(1-\frac{1}{2}\mathrm{tr}U_\Box) \ . \tag{3}$$

Here the last sum is over elementary plaquettes bounded by 4 links and $U_\Box$ denotes the path-ordered product of the 4 gauge elements along the boundary of the plaquette $\Box$. The last term, the potential term, is automatically bounded and for a given finite total energy the same is the case for the first term, the kinetic term. Thus the phase space corresponding to a given energy-surface is compact.

The compactness of the phase space implies that the spectrum of Lyapunov exponents (which we overall will assume to be well defined quantities for the lattice system) is independent of the choice of norm on the space of field configurations. The measures of distances between lattice field configurations $\mathbf{X}(t)$ and $\tilde{\mathbf{X}}(t)$ which are employed in [14] - [19] are semi-positive and, typically, of the form (cf. e.g. [18])

$$\mathcal{D}(\mathbf{X},\tilde{\mathbf{X}}) \sim \frac{1}{N^3}\ (\ \sum_{i\in\Lambda}|tr(P_iP_i^\dagger) - tr(\tilde{P}_i\tilde{P}_i^\dagger)|\ +\ \sum_{\Box}|trU_\Box - tr\tilde{U}_\Box|\ ) \tag{4}$$

which in the limit as the lattice constant $a \to 0$ measures the (average) local differences in the electric and magnetic field energy,

$$\mathcal{D}((E,B),(\tilde{E},\tilde{B})) \sim \frac{1}{V}\int d^3x\ (|E^2-\tilde{E}^2| + |B^2-\tilde{B}^2|)\ \ . \tag{5}$$

Gauge equivalent field configurations have a vanishing distance since the distance measure (4) is gauge invariant.[1] As discussed at the end of the appendix such a choice of distance will not affect the calculation of the largest Lyapunov exponent.

It is easy to see that if $\mathbf{X}^{(a=1)}(t)$ solves the Hamiltonian equations for $a=1$ then $\mathbf{X}^{(a)}(t) = \mathbf{X}^{(a=1)}(t/a)$ solves the same equations for general a and this time-scaling implies the following scaling of the maximal Lyapunov exponent :[2]

$$\begin{aligned}\lambda_{max}(a) &\equiv \lim_{t\to\infty}\frac{1}{t}\log(\frac{\|\delta\mathbf{X}^{(a)}(t)\|}{\|\delta\mathbf{X}^{(a)}(0)\|}) \\ &= \frac{1}{a}\lim_{t/a\to\infty}\frac{1}{(t/a)}\log(\frac{\|\delta\mathbf{X}^{(a=1)}(t/a)\|}{\|\delta\mathbf{X}^{(a=1)}(0)\|}) = \frac{\lambda_{max}(a=1)}{a}\end{aligned} \tag{6}$$

[1] As regards gauge invariance in the resulting measures of chaos, we note that since "space-time" is not involved in the gauge transformations, we expect Lyapunov exponents and correlation lengths to be gauge invariant measures of temporal and spatial chaos in the context of Yang-Mills fields, whereas such gauge invariant measures of chaos are more difficult to construct for a typical metric in general relativity [27].

[2] In the equation (6) we should in fact replace the small deviation vector $\delta\mathbf{X}$ by a tangent vector to the manifold, cf. the appendix.

Equation (6) implies that $a\lambda_{max}(a)$ is invariant under rescaling of a. If we assume ergodicity over the energy surface (modulo the known constraints) we see that in particular, $a\lambda_{max}(a)$ is uniquely determined by the energy $aH(a) = H(a = 1)$ or equivalently when lattice size N^3 is fixed it has to be some function of the energy per plaquette $aE(a) = E(a = 1)$,

$$a\ \lambda_{max}(a, E(a)) = f\ (a\ E(a)) \tag{7}$$

Such a functional relationship even persists in the thermodynamic limit, $N \to \infty$, for any fixed values of a and $E(a)$. Thus in order to study the dependence of the maximal Lyapunov exponent with energy density, it is sufficient to consider the equations of motion for a fixed value of the lattice constant a, e.g. $a = 1$, as a function of energy density ($\propto$ energy/plaquette) and rescale the results back afterwards.

Studying the classical dynamics in time rather than the ensemble distribution makes it possible to extract dynamical information as well as thermodynamic information. By measuring the maximal Lyapunov exponent $\lambda_{max}(a)$ for some fixed lattice spacing a of the system as a function of the energy, one may also hope to gain insight into the spatial correlations of the fields. Although a precise relationship has not been established yet, it seems plausible that when propagation of information to nearest neighbors occurs at a fixed speed, a small value of λ_{max} should lead to a larger correlation length than a larger value of λ_{max}. The reason is that in order for the dynamics to create long range spatial correlations in the system, information has to propagate for a long time (= distance) without seriously being attenuated by the chaotic behavior, the 'strength' of which is reflected in the value of λ_{max}. Clearly, it deserves further investigation to establish a precise relationship between λ_{max} and the spatial correlation length ξ (in lattice units). Such investigations would be of considerable interest also in the more general case where the non-Abelian gauge fields are given a non-zero mass, with an adjustable mass parameter, by coupling to a Higgs-field. In our case, which involves massless non-Abelian gauge fields, we expect that it is in the limit as λ_{max} goes to zero that the spatial correlation length ξ (in lattice units) will diverge. [3]

In fact, a complicated question (which needs further study) is what we could mean by a "continuum limit" for a classical, lattice regularized non-Abelian gauge field theory. Whereas the issue of extracting expectation values of observables in the "continuum limit" is standard in interpreting simulations of quantum field theories in the imaginary time (Euclidean) formalism, the issue of extracting "continuum results" (rather than lattice artifacts) is much less well-known in the case of real-time simulations of classical non-Abelian gauge theories. Moreover, it is an important question of interest in physics, since not many non-perturbative

[3] It is a generic expectation for a chaotic, non-integrable, spatially extended system (with a spatial propagation of disturbances at some fixed speed) that increasing the temporal chaos, as captured e.g. by measuring the maximal Lyapunov exponent λ_{max}, will be connected to more spatial chaos, as measured, say, by an inverse spatial correlation length $\sim \xi^{-1}$. Cf. e.g. discussions in [24, 25]. Note, however, that an integrable, spatially extended Hamiltonian system can also have finite spatial correlations due to the initial conditions rather than the dynamics. I.e., an integrable Hamiltonian spatially extended system will not be able to establish spatial correlations in field configurations which initially are spatially de-correlated.

methods are available to study the time evolution of quantum field theories, such as Q.C.D., or the weak sector of the Standard Model, and one almost has to resolve to study the real time evolution of the classical fields - an approach approximately justified when probing physical situations (e.g. at high temperatures) where the quantum fields are expected to behave semi-classically (see also, e.g. discussions in [23]).

For the study of the time evolution of Yang-Mills fields which initially are far from an equilibrium situation, the (classical) field modes will exhibit a never ending dynamical cascade towards the ultraviolet and after a certain transient time, the cut-off provided by the spatial lattice will prevent the lattice gauge theory from simulating this cascade. It is therefore immediately clear that the lattice regularized, classical fields will not approach a "continuum limit" in the sense of simulating the dynamical behavior of the classical continuum fields in the $t \to \infty$ limit (we may also say that the continuum classical theory does not exist in the infinite time limit).

We have several different forms of lattice artifacts in the lattice simulation of real-time dynamical behavior of the continuum classical Yang-Mills fields:

(1) Lattice artifacts due to the compactness of the group. The magnetic term (the second term) in the Kogut-Susskind Hamiltonian (3) is uniformly bounded, $0 \leq 1 - \frac{1}{2} Tr U_{\Box} \leq 2$, due to the $SU(2)$ compactification. Thinking in terms of statistical mechanics for our classical lattice system, we expect that after some time the typical field configuration has equally much energy in all modes of vibration - independent of the frequency[4] - and the total amplitude of the classical field, and the energy per lattice plaquette, is thus small for a fixed low energy. For low energy, when the average energy per plaquette is small, the lattice artifacts due to the compactness of the gauge group are thus negligible.

(2) For small energy per plaquette, we thus expect that the dominant form for lattice artifact is due to the fact that an appreciable amount of the activity (for example the energy) is in the field modes with wavelengths comparable to the lattice constant a. This short wavelength activity at lattice cut-off scales is unavoidable in the limit of long time simulation of an initially smooth field configuration (relative to the lattice spacing), or already after a short time if we initially have an irregular field configuration.

We conclude that for the simulation of gauge fields far from equilibrium, we will have the best "continuum limit" if we simulate, for a short period of time, an initial smooth ansatz for the fields in the region of low energy per plaquette.[5]

[4] This situation is very different in the quantum case. Planck's constant $\hbar$ introduces a relation $E = \hbar\omega$ between the energy of a mode of vibration and its frequency, implying that a mode with a high frequency also has a high energy. With a given available finite total energy, modes with high frequencies will therefore be suppressed. Quantum mechanically, we thus have that at low energy only excitations of the longest wavelengths appear.

[5] We are here imagining a situation where the spatial correlations in the monitored field variables are so large that they lose memory of the underlying lattice structure (including the lattice spacing a). In the extreme opposite limit, one could imagine models in situations with random fluctuating fields on the scale of the lattice constant, i.e. with (almost) no spatial correlations from link to link. If the field variables fluctuate independent of each other (independent of their neighbors) one could imagine the model to be

This is if we have no quantum mechanics. However, Yang-Mills fields exist as quantum theories. Thus, the interesting physical definition of a "continuum limit" of the classical lattice gauge theory, is to identify regions in the parameter space for the classical lattice gauge theory which, for (shorter or longer) intervals of time, probe the behavior of the time evolution of semi-classical initial configurations (with many quanta) of the quantum theory, for example Q.C.D. (See also e.g. discussion in [23]).

There is by now a large amount of literature which reports investigations of temporal chaos on the lattice gauge theory measured by the spectrum of Lyapunov exponents (either the principal Lyapunov exponent or the entire spectrum of Lyapunov exponents) and how Lyapunov exponents depend on the average energy per plaquette, E, of the lattice field theory. We refer to a sequence of articles by Müller et al [14], Gong [16], Biró et al. [17] and a recent book by Biró et al [19] which present numerical analyses of the classical $SU(2)$ lattice gauge model. Their numerical results provide evidence that the maximal Lyapunov exponent is a monotonically increasing continuous function of the scale free energy/plaquette with the value zero at zero energy.

Müller et al. [14] report a particular interesting interpretation of numerical results for the dynamics on the lattice, namely that there is a linear scaling relation between the scale free maximal Lyapunov exponent, $\lambda_{max}(a=1)$ and the energy per plaquette $E(a=1)$. The possible physical relevance of this result is seen when we rescale back to a variable lattice spacing a and note that the observed relationship is in fact a graph of $a\lambda_{max}(a)$ as a function of $aE(a)$. Thus being linear, cancellation of a factor a implies that $\lambda_{max}(a) = \text{const} \times E(a)$ and thus there is a continuum limit $a \to 0$ either of both sides simultaneously or of none of them. In the particular case where the energy per mode ($\propto$ energy per plaquette) is taken to be a fixed temperature T (cf. [21]), one deduces that the maximal Lyapunov exponent has a continuum limit in real time.

We shall here, however, note that the linear relationship found between λ_{max} and E is based on a graph of their interdependence (indicating that such a scaling relation exists) in the approximate scaling region (for $a = 1/2$) of E between 0.6 and 4, where the maximal Lyapunov exponent grows from 0.2 to 1.3 (cf. figure 1) :

$$\lambda_{max} \approx 0.32\, E \ . \tag{8}$$

We shall argue that the apparent linear scaling relation is a transient phenomena residing in a region extending at most a decade between two scaling regions, namely for small energies where the Lyapunov exponent scales with an exponent which could be close to 1/4 and a high energy region where the scaling exponent is at most zero:

$$\beta = \frac{d\log\lambda}{d\log E} = \begin{cases} \sim 1/4 & \text{for } E \to 0 \\ \leq 0 & \text{for } E \to \infty \end{cases} \tag{9}$$

invariant (with respect to the monitoring of many variables) under changes of the lattice spacing, a. Thus, it appears that lattice cut-off independence of numerical results can not be a sufficient criterion for the results to report "continuum physics".

In fact, we believe that the apparent linear scaling region, equation (8), is a complete artifact of the $SU(2)$ compactification on a lattice. We argue by giving a newly obtained [28] rigorous upper bound for the maximal Lyapunov exponent, as well as plausible lower bounds obtained from general scaling arguments [26] of the continuum classical Yang-Mills equations in accordance with simulations on homogeneous models [4] and consistent with the figures in [14] and [17]. Thus the existence of the two scaling regions, equation (9), is established by a combination of numerical evidence and analytical arguments. In figure 1 (a) we have plotted the numerical results reported in Müller et al [14] which exhibits a systematic deviation from a straight line. A recent numerical investigation by Krasnitz (cf. [20], figure 3) has independently confirmed such a systematic deviation. According to Krasnitz statistically significant deviations from the straight line is as large as 10 %. In figure 1 (b) we have made a log-log plot of the results obtained by Müller et al [14].

Regarding the limit for high energies per plaquette a rigorous result [28] (a sketch is shown in Appendix A) shows that the $SU(2)$ scale free ($a = 1$) lattice Hamiltonian in d spatial dimensions has an upper bound for the maximal Lyapunov exponent

$$\lambda_B = \sqrt{(d-1)(4+\sqrt{17})} \tag{10}$$

which for $d = 3$ becomes $\lambda_B = 4.03...$ This result is arrived at by constructing an appropriate norm on the phase space and showing that the time derivative of this norm can be bounded by a constant times the norm itself, hence giving us an upper bound as to how exponentially fast the norm can grow in time. The upper bound (10) shows that a linear scaling region, i.e. a constant $\beta = d\log\lambda/d\log E$, can not extend further than around $E \sim 10$ on the figure 1. Beyond that point the maximal Lyapunov exponent either saturates and scales with energy with an exponent which approaches zero or it may even decrease over a region of high energies, yielding a negative exponent[6]. It should be noted that the upper bound (10) is independent of the lattice size and the energy (but scales with $1/a$).

In the opposite limit, the "continuum limit", where the average energy per plaquette $E \to 0$, the finite size of the lattice makes it much more difficult to analyze the behavior of the gauge fields and the principal Lyapunov exponent on the lattice. However, we shall argue that the scaling exponent of the Lyapunov exponent more likely will be closer to $\sim 1/4$ than the scaling exponent ~ 1 observed in the intermediate energy region $1/2 \leq E \leq 4$.

In figure 1 (b) we have made a log-log plot of the results obtained by Müller et al. [14]. It appears that for $E \sim 1/2$ there is a cross-over to another scaling region. Although the data points in this region are determined with some numerical uncertainty [15] we note that they are consistent with a scaling with exponent $\sim 1/4$. Moreover, it is a numerically established fact that the homogeneous Yang-Mills equations have a non-zero Lyapunov exponent which by elementary scaling arguments scales with the fourth root of the energy density and has the approximate form [4]:

$$\lambda_{\max} \approx 0.38\ E^{1/4}\ . \tag{11}$$

[6] Some very heuristic arguments yields $\beta = -1/2$ asymptotically.

On the lattice the above scaling relation is valid for spatially homogeneous fields, i.e. the maximal Lyapunov exponent scales with the fourth root of the energy per plaquette. By continuity, fields which are almost homogeneous[7] on the lattice will, in their transient, initial dynamical behavior, exhibit a scaling exponent close to 1/4. In fact, the same scaling exponent[8] would also hold for the inhomogeneous Yang-Mills equations [26] had the fields been smooth relative to the lattice scale, so derivatives, ∂_μ, are well approximated by their lattice equivalent and we are allowed to scale lengths as well. This is seen from scaling arguments for the continuum Yang-Mills equations :

$$D_\mu F^{\mu\nu} = 0 \quad , \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu] \quad , \quad D_\mu = \partial_\mu - ig[A_\mu, \quad]$$

Not taking boundary conditions into account, these equations are invariant when ∂_μ and A_μ are scaled with the same factor α. That is, if $A(x,t)$ is a solution to the equations, then $\frac{1}{\alpha}A(\alpha x, \alpha t)$ is also a solution. The energy density E, which is quadratic in the Yang-Mills field curvature tensor $F^{\mu\nu}$, then scales with α^4. The same arguments as leading to equation (6) then show that if we perform a measurement of the maximal Lyapunov exponent λ_{max} over a time short enough for the solutions to stay smooth, then λ_{max} scales with $E^{1/4}$, as was also observed by Müller et al. p. 3389 in [14]. These scaling arguments do not carry over to infinite time averages since solutions on the lattice tend to be irregular. We believe, however, that the cross-over observed in figure 1(b) is a feature of the Kogut-Susskind model and not just a finite size effect, in particular since there is evidence [22] that the correlation length for energies $E \sim 1/2$ is of the order of a few lattice units only. As the numerical simulations of Müller et al. are performed on lattices $\sim 20^3$ and since the calculation of Lyapunov exponents is a local calculation (when considering infinitesimal deviations), a finite size effect should show up for energies somewhat below $E \sim 1/2$.

The expectations, arrived at here, are supported by an even more striking result, which is the numerical simulations of Biró et al ([17], figure 12) for the Kogut-Susskind lattice model with a $U(1)$ group showing (see figure 2) a steep increase of the maximal Lyapunov exponent with energy/plaquette in the interval $1 \leq E \leq 4$. The continuum theory here corresponds to the classical electromagnetic fields which have no self-interaction and thus the Lyapunov exponent in this limit should vanish. The discrepancies in this case were in [17] attributed to a combined effect of the discreteness of the lattice and the compactness of the gauge group $U(1)$ and were not connected with finite size effects. This suggests strongly that in the case of $SU(2)$ a similar cross-over around $E \sim 1/2$ should have the same origin and likewise, not be seen as a finite size effect. In particular, it makes it difficult to believe that we in the case of numerical studies of a $SU(2)$ Kogut-Susskind Hamiltonian system can base continuum physics on results from simulations in the same interval of energies where the

[7]Note that such field configurations are (ungeneric) examples of field configurations which exhibit correlation lengths much larger than the lattice spacing.

[8]In the lattice simulations the energy of a given plaquette is a fluctuating quantity and since the function $E^{1/4}$ is convex one would expect the coefficient of proportionality to be somewhat smaller than in formula (11). Numerically the coefficient turns out to be around half of the above value, cf. figure 1 (a-b).

$U(1)$ simulations fail to display continuum physics. On the contrary, we suspect that what could reasonably be called "continuum physics" has to be extracted from investigations of the Kogut-Susskind lattice simulations for energies per plaquette which are at least smaller than $E \sim 1/2$. In conclusion, we believe that there is indeed a cross-over around $E \sim 1/2$ but we do not attribute this to finite size effects[9] which would invalidate the physical relevance of the lower region. Rather, we believe and conjecture that this phenomena will persist even for infinitely big lattices still showing a cross-over around $E \sim 1/2$, below which we approach "continuum physics" and above which discreteness of the lattice and compactness of the gauge group are no longer negligible.

There is quite a simple intuitive explanation for the kind of transition taking place for energies in the region above 1/2 for the $SU(2)$ lattice gauge model and for the saturation of the Lyapunov exponent in the regime for high energy per plaquette: The Hamiltonian (3) consists of two terms, the first one is a kinetic energy term (electric term) which is a quadratic form in the momenta, the second is a potential term (magnetic term) which is uniformly bounded $0 \leq 1 - \frac{1}{2} Tr U_\Box \leq 2$, due to the $SU(2)$ compactification. For small energies per plaquette, $E \ll 1$, both terms will be of comparable size. In fact, as energy increases from zero, the potential energy increases too, but as long as the energy/plaquette is much smaller than 1, the gauge holonomy $U_\Box$ calculated around a Wilson loop is close to the identity so that field curvatures are small. The dynamics is then essentially confined to the Lie algebra, i.e. the tangent plane of the identity element in $SU(2)$. However, since the Lie-group itself is curved, higher order non-linear terms become important as the energy per plaquette increases further. Although this does not imply 'more chaos', it does suggest a steeper increase in the Lyapunov exponent. As energy increases even more, $E \gg 1$, the $U_\Box$ will experience the finiteness of the group $SU(2)$ and energy will only be pumped into the kinetic term. We note that if only the kinetic term had been present in the Kogut-Susskind Hamiltonian (3), the system would be integrable. Since the potential term only provides a uniformly bounded perturbation of the dynamics, it is reasonable to expect (and shown in the appendix) that the spectrum of Lyapunov exponents will saturate as the energy per plaquette increases for the Kogut-Susskind model. The chaos generated by the non-linear potential energy term does not increase, only the energy in the kinetic energy (the electric fields) increases.

Finally, we will give a few comments on possible implications of the scaling relation (9) proposed in this investigation. Since by intrinsic scaling arguments (cf. equation (7)) one has a functional relation between $a\lambda_{max}(a)$ and $aE(a)$, a 1/4 scaling for small energies would imply that $\lambda(a) \propto a^{-3/4} E(a)^{1/4}$, in which case one cannot achieve a continuum limit simultaneously for the maximal Lyapunov exponent and the temperature (assuming that it is proportional to the average energy per plaquette $E(a)$, cf. [21]), in particular, the former would be divergent if the temperature is kept fixed. There is no particular contradiction in

[9] It would be valuable to study the spatial correlation lengths on the lattice system for energies per plaquette in the interval $0 \leq E \leq 1$, i.e. for an interval of energies which includes the apparent cross-over at $E \sim 1/2$.

this statement, however, as there is, a priori, no need for having a finite Lyapunov exponent in the continuum limit. The erratic and fluctuating behavior of the fields one expects in time as well as in space (for numerical evidence, cf. also [10]) on very small scales indicates that a Lyapunov exponent will not be well defined. Clearly, this question deserves further investigation and we do believe that it is of considerable interest to understand in which sense we can or cannot have a "continuum limit" of a spectrum of Lyapunov exponents for a (non-dissipative) field theory like the $SU(2)$ Yang-Mills theory when we study, say, the dynamical behavior in a given finite (physical) volume $\sim L^3 \sim (Na)^3$ with periodic boundary conditions in the limit of long time $t \to \infty$. A question of equal interest which may be related to the question above is to understand the (physical) spatial correlations in the system. Studies of (semi)classical dynamically generated temporal and spatial chaos may be important for the understanding of randomness in space and time of field configurations in Q.C.D. (see also discussions in e.g. [19, 26]). The present investigation has been devoted to a discussion of a possible "continuum limit" of (chaotic aspects of) purely *classical* Yang-Mills fields regularized on a spatial lattice, whereas Yang-Mills fields in Nature are implemented as quantum fields, regularized by $\hbar$ (supplemented by a regularization/renormalization prescription). A connection between (semi)classical aspects of the real quantum theory and the behavior of the fields in the lattice regularized classical field theory would be most interesting to establish on a more rigorous basis.

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Appendix

We shall briefly sketch the proof that for an $SU(2)$ lattice Hamiltonian model (3) with a fixed lattice constant ($a = 1$), the maximal Lyapunov exponent is uniformly bounded from above, independent of the energy and the lattice size. We refer to [28] for details in the proof.

First we shall illustrate the main trick which is quite trivial: Consider a harmonic oscillator whose evolution is governed by the equations $\dot{x} = y$ and $\dot{y} = -\omega^2 x$. This system is integrable and thus it has vanishing Lyapunov exponents. Now, a way to show this is to consider the norm on the tangent space (which here is the same as the space itself) $\|(x, y)\|^2 = \omega^2 x^2 + y^2$ which for ω non-zero is equivalent to any other norm on R^2. The time-derivative of this norm is given by $\frac{d}{dt}\|(x, y)\|^2 = 2\omega^2 xy + 2(-\omega^2 x)y = 0$ and hence the norm is invariant under the flow. It follows that all Lyapunov exponents vanish.

In the case of $SU(2)$ lattice gauge dynamics, the momentum P will play the role of ω above and thus be part of an integrable dynamics (physically one may think of a rotor on a sphere). From this integrable part, we shall construct a 'good' metric for which norms of vectors in the tangent space would be constant in time, had it not been for the non-linear

coupling through the plaquettes. Taking the time derivative of the norm, we get a variety of terms which, due to the compactness of $SU(2)$ and hence uniform boundedness of the couplings, can be bounded by our 'good' metric itself ! This kind of bootstrap argument will give us a uniform upper bound as to how fast the norm can grow, and from this we deduce rigorous bounds on the maximal Lyapunov exponent.

The Lie group $SU(2)$ will be considered in the quaternion representation, which is a four dimensional embedding using Pauli matrices (not explicitly shown here) for which one writes :

$$U = \begin{pmatrix} u_0 + iu_3, & iu_1 + u_2 \\ iu_1 - u_2, & u_0 - iu_3 \end{pmatrix} \equiv (u_0, u_1, u_2, u_3)$$

and define the usual Euclidean inner product in R^4 by :

$$\langle U, V \rangle \equiv u_0 v_0 + u_1 v_1 + u_2 v_2 + u_3 v_3 = \frac{1}{2} \mathrm{tr}(U V^\dagger) \ .$$

This yields a representation of the 6 dimensional tangent manifold $T\ SU(2)$, such that a base point $U \in SU(2)$ and a tangent vector $P \in T_U SU(2)$ are given coordinates $(U, P) \in R^4 \oplus R^4$, subjected to the constraints :

$$\langle U, U \rangle \equiv 1 \quad \text{and} \quad \langle P, U \rangle \equiv 0 \ .$$

The entire phase space is denoted

$$M = \prod_\Lambda T\ SU(2) \ .$$

Using the above notation, the Hamiltonian (3) can be written as follows :

$$H = \sum_{i \in \Lambda} \frac{1}{2} \langle P_i, P_i \rangle + \sum_\Box (1 - \frac{1}{2} \mathrm{tr} U_\Box) \ ,$$

where the lattice constant is set to $a = 1$. The variation of H with respect to U_i gives us $\delta H = \langle \delta U_i, V_i \rangle$ and thus a force acting upon U_i :

$$V_i = \sum_{(jkl)} U_j U_k U_l^\dagger \ .$$

The sum above extends over all neighboring links, such that $(ijkl)$ forms a plaquette. Hence if the dimension of the space is d, V_i depends in total of $2(d-1)(4-1) = 6(d-1)$ neighboring links but not on U_i itself.

Under the constraints on U and P given above, the Hamiltonian gives rise to a flow $\phi^t : M \to M$, $t \in R$ determined by the following differential equations (see e.g. [19]) :

$$\begin{aligned} \dot{U}_i &= P_i \ , \\ \dot{P}_i &= V_i - U_i \langle U_i, V_i \rangle - U_i \langle P_i, P_i \rangle \ . \end{aligned} \qquad (12)$$

Consider a small deviation $\delta\mathbf{X}$ of a trajectory

$$\mathbf{X} + \delta\mathbf{X} = \{U_i + \delta U_i, P_i + \delta P_i\}_{i\in\Lambda} \in M$$

and linearize the above equation (12) in $\delta\mathbf{X}$ to obtain the evolution now for a tangent vector which we again, by slight abuse of notation, denote $\delta\mathbf{X} = \{x_i, y_i\}_{i\in\Lambda} \in T_{\mathbf{X}}M$

$$\begin{aligned} \dot{x}_i &= y_i \ , \\ \dot{y}_i &= v_i - U_i\langle U_i, v_i\rangle - x_i\langle U_i, V_i\rangle \\ &- U_i\langle V_i, x_i\rangle - x_i\langle P_i, P_i\rangle - 2U_i\langle P_i, y_i\rangle \ . \end{aligned}$$

A 'good' norm is now given by the following Riemann metric :

$$G(\delta\mathbf{X}, \delta\mathbf{X}) = \frac{1}{2}\sum_{i\in\Lambda}\left(\langle x_i, x_i\rangle(C + \langle P_i, P_i\rangle) + \langle y_i, y_i\rangle - \langle x_i, P_i\rangle^2 - \langle y_i, U_i\rangle^2\right) \ . \tag{13}$$

with an optimal choice of $C = 2\sqrt{17}(d-1)$. For any fixed energy, H, $\langle P_i, P_i\rangle$ is bounded from above by $2H$, and a small calculation shows that G is positive definite as a quadratic form on TM. As we have noted earlier, our phase space is compact for any fixed energy and it follows that $\sqrt{G}$ provides a norm which is equivalent to any other norm on TM. If $\|\delta\mathbf{X}\|$ is another norm, compactness implies the existence of $k_2, k_1 > 0$ such that :

$$k_1\|\delta\mathbf{X}\| \leq \sqrt{G(\delta\mathbf{X}, \delta\mathbf{X})} \leq k_2\|\delta\mathbf{X}\| \ . \tag{14}$$

In particular, the maximal Lyapunov exponent measured with any two equivalent norms are identical. From (14) we have :

$$\lim_{t\to\infty}\frac{1}{t}\ \log\frac{\|\delta\mathbf{X}[t]\|}{\|\delta\mathbf{X}[0]\|} = \lim_{t\to\infty}\frac{1}{2t}\ \log\frac{G((\delta\mathbf{X}, \delta\mathbf{X})[t])}{G((\delta\mathbf{X}, \delta\mathbf{X})[0])} \ . \tag{15}$$

Now, using the constraints above, one verifies that the derivative of G along the flow equals :

$$\begin{aligned} \frac{d}{dt}G(\delta\mathbf{X}, \delta\mathbf{X}) &= \sum_i \langle x_i, y_i\rangle(C - \langle U_i, V_i\rangle) + \langle x_i, x_i\rangle\langle P_i, V_i\rangle \\ &- \langle x_i, P_i\rangle\langle V_i, x_i\rangle + \langle v_i, y_i\rangle - \langle v_i, U_i\rangle\langle U_i, y_i\rangle \ . \end{aligned}$$

The v_i entering the equations above is the tangent vector associated with V_i and hence, it is a sum of $6(d-1)$ terms of the type $x_j U_k^\dagger U_l^\dagger$ involving the tangent vectors x_j of the neighboring lattice sites (cf. above). A tedious but straight-forward calculation shows that the above time derivative satisfies :

$$|\frac{d}{dt}G(\delta\mathbf{X}, \delta\mathbf{X})| \leq 2\lambda_B G(\delta\mathbf{X}, \delta\mathbf{X})$$

with $\lambda_B = \sqrt{d-1}\sqrt{1 + \sqrt{1 + \frac{1}{16}}}$. One thus has the bound on the growth rate of the norm :

$$\sqrt{G(\delta\mathbf{X}[t], \delta\mathbf{X}[t])} \leq \exp(\lambda_B t)\sqrt{G(\delta\mathbf{X}[0], \delta\mathbf{X}[0])} \ .$$

In the case of $d = 3$, we have $\lambda_B = 4.03...$ which therefore gives an upper bound to the maximal Lyapunov exponent, independent of the energy and the lattice size.

The metric we have constructed above is not gauge invariant and in principle one would like to study the dynamics and the metric of the associated moduli space, i.e. the space M modulo gauge transformations. However, in the study of upper bounds for Lyapunov exponents, this distinction is not necessary since gauge transformations are symmetry transformations of the dynamics and hence the associated Lyapunov exponents vanish. This is nicely illustrated by Gong in [16], where the complete Lyapunov spectrum is calculated for 2^3 and 3^3 lattices. One third of the Lyapunov exponents vanish due to the gauge symmetry, expressed by the Gauss law constraint and residual static gauge transformations. A compactness argument again shows that using a gauge invariant distance measure (like the one used in [14, 18]) which is only semi-positive but which distinguishes gauge non-equivalent configurations will yield the same measurements of Lyapunov exponents as a strictly positive metric.

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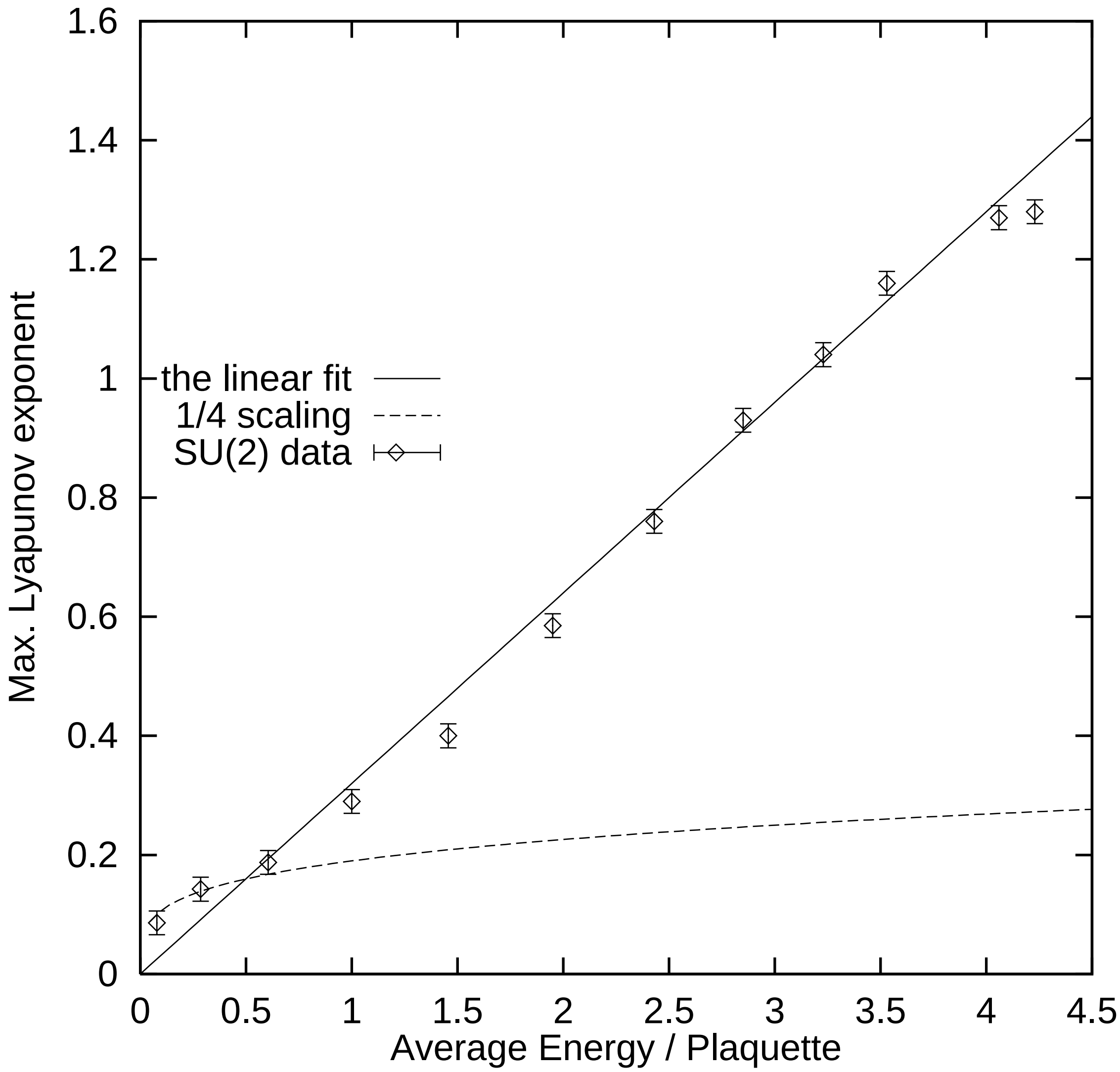
Figure captions :

Figure 1(a) shows for $SU(2)$ a plot of the maximal Lyapunov exponent as a function of the average energy per plaquette. The data points (diamonds) are adapted from Müller et al. [14] (see also Biró et al. [19], p. 192) where results are obtained from a simulation on a 20^3 lattice with periodic boundary conditions in all spatial directions. The solid line is a linear fit through the origin and the dotted line is the function $\frac{1}{2} \times 0.38E^{1/4}$ which is half of the result (cf. formula (11)) obtained from the homogeneous case (see also figure 1 (b)).

Figure 1(b) displays a log-log plot of the same data points as in figure 1(a). On the figure we also show the (theoretically obtained) upper bound for the maximal Lyapunov exponent.

Figure 2(a) shows a plot similar to figure 1(a) for $U(1)$. The data points are adapted from Biró et al. [17]. We note, that the Lyapunov exponent grows, as a lattice artifact, in exactly the same region of energy/plaquette $1 \leq E \leq 4$ where the figure 1(a) (or 1(b)) is reported to exhibit continuum physics.



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0
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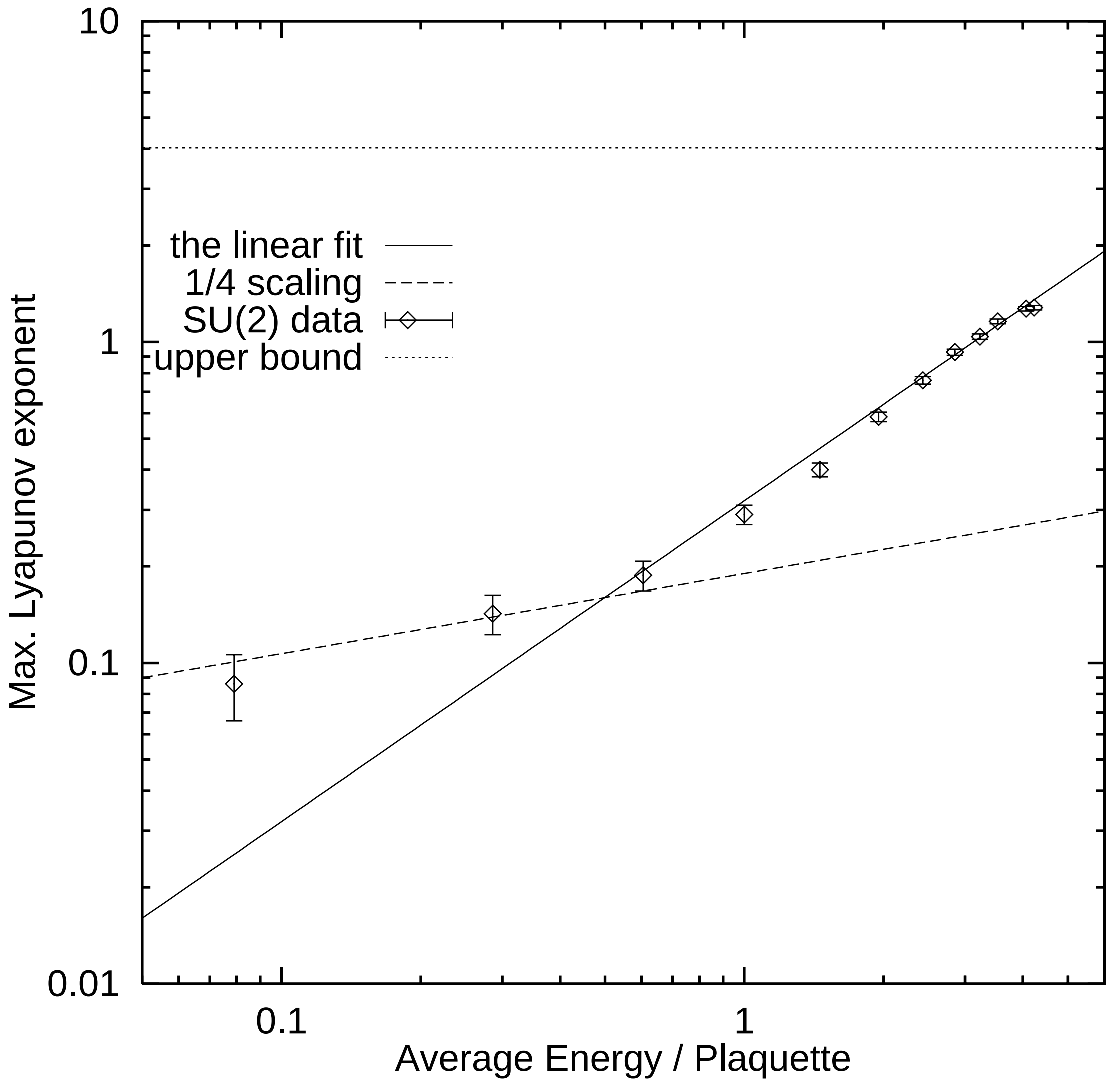
3
3.5
4
4.5
Max. Lyapunov exponent
Average Energy / Plaquette
the linear fit
1/4 scaling
SU(2) data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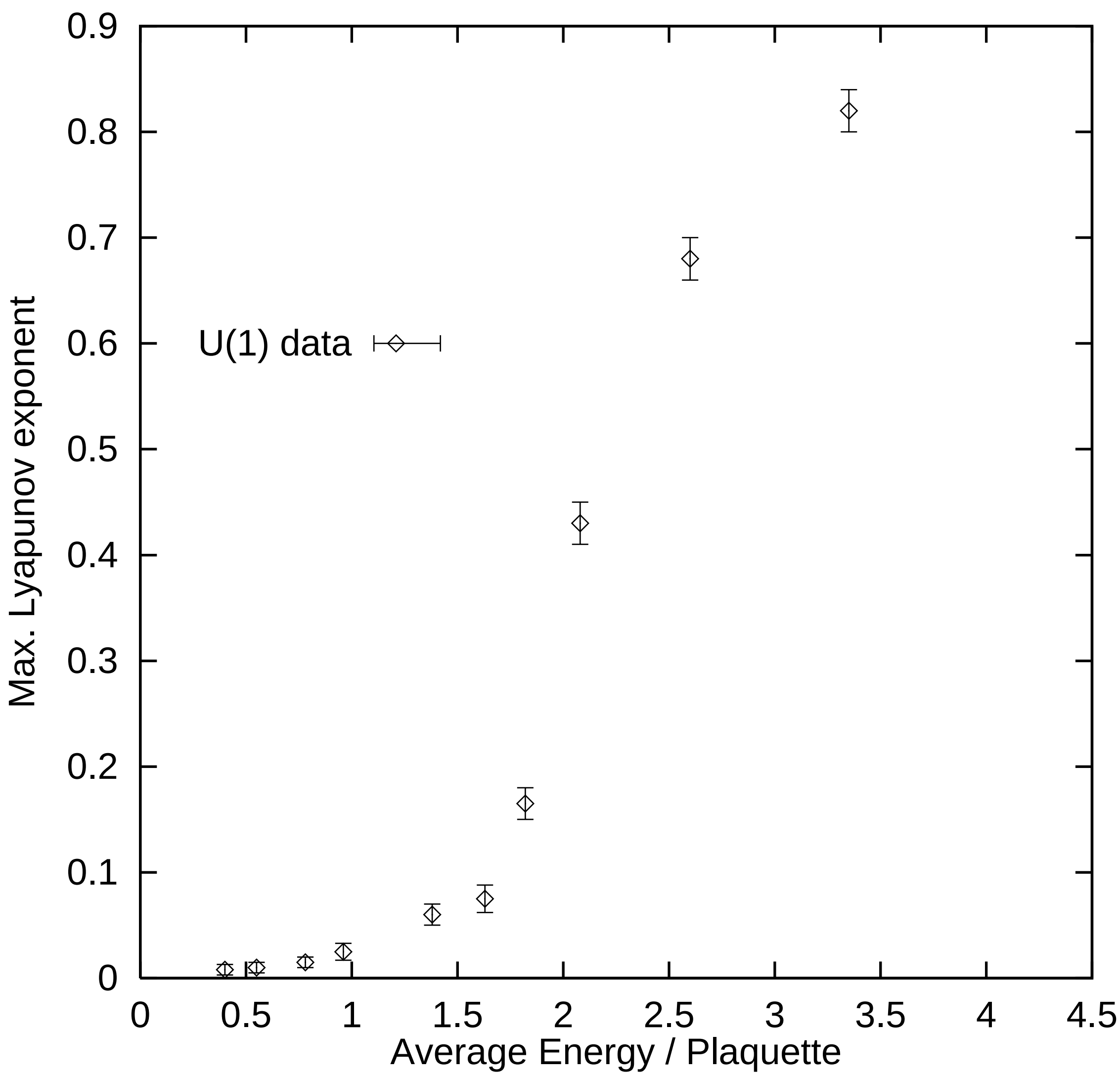
0.1
1
Average Energy / Plaquette
Max. Lyapunov exponent
the linear fit
1/4 scaling
SU(2) data
upper bound



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1
1.5
2
2.5
3
3.5
4
4.5
Max. Lyapunov exponent
Average Energy / Plaquette
U(1) data